



FREE VIBRATIONAL EFFECT OF FOOT BRIDGE WITH COMPRESSIVE FORCES

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ABSTRACT

This paper examines the free vibrational effect of a footbridge with compressive forces. The governing partial differential equation were analysed to determine the behaviour of the system under consideration. The methods of series solution and numerical method were used to solve the governing equation. Result revealed there is no noticeable effect of damping in the system as the motion of deflection remains constant for all values of the damping coefficient. Also, the lateral displacement increases for various values of spring constant

Keywords: *Foot Bridge Free Vibration, Series Solution, Compressive Force*

INTRODUCTION

Footbridges are fundamental structural elements of many engineering application and have extensively studied in connection with machine processing, guide-way system and design across railway, highway and so on (Akinpelu, 2012). Amin *et al.*, (2008). A footbridge are placed on beam of an elastic foundation such as plate. In other words, it is an elastic medium that constitute part of any machinery for isolation purposes, concrete structure on soil (Blanchard *et al.*, 1977; Bruno *et al.*, 2011).

Footbridges are mainly used as a basis of supporting the movement of people against traffic, accident on the highway (Vu *et al.*, 2000). Without the knowledge of footbridge beam theory, these would have not been possible to manufacture of such structure. The crowds walking on footbridge impose to the structure of a Dynamic Lateral Excitation at a close frequency (Chen and Robertson, 1972; Chen, 2002; Dallard *et al.*, 2001). When the first mode of lateral vibration of a footbridge falls in the same frequency interval, then the resonance phenomena is activated, the oscillation amplitude increase and the pedestrian are forced to change their way of walking up to the so-called structure pedestrian synchronization occurring if the oscillation amplitude is enough (Fibya, 1972; Fujino *et al.*, 1993; Gbolagade *et al.*, 2003). This phenomena has been often experimentally detected also analyzed in several studies Dallard *et al.*, (2001). Lijun *et al.* (2015), investigated the lateral vibration by employing the center manifold theory and first-order approximation solution of the periodic vibration of the footbridge using the energy method (Haun, 2018; Hsu, 2008; Izebela, 2010; Usman, 2003). Numerical simulation is carryout to verify the validity of analytical expression which reduce the amplitude and frequency of the footbridge especially for a large amplitude case (Jung-Cheng, 2008; Mahmoud, 2013; McRobie *et al.*, 2003; Mehri and Rahmani, 2009).

Brownjohn *et al.*, (2004) investigated the load induced by the pedestrian on footbridges which was tacked progressively by estimation of the force originated by single walker the result were further extrapolated to the case of a crowd moving on the footbridge using the expression of Fourier decomposition method (Newland, 2003; Ondreji and Jiri 2013; Pavic and Reynold, 2005; Ronnquist, 2005).

In this work, we aim to analyze the crowd effect on lateral motion of footbridge under crowd- load- ing supported beam on elastic foundation. Governing partial differential equation of beam theory can be solved by several methods such as finite difference method, differential quadrature element method, differential transformation method, Fourier harmonic decomposition method, Fourier series method and so on, but in this study, the use of series solution method in solving the governing partial differential equation was considered. The objectives of this work are: to present a technique for determining the response of

simply supported Euler-Bernoulli equation subjected to crowd loading; to obtain the analytical solution of the governing partial differential equations of the beam; to determine the lateral displacement of the beam as a function of time and space $U(x,t)$ subjected to initial and bounary conditions of the system; to use the graphical analysis of the numerical result to show responses of lateral vibration.

MATHEMATICAL FORMULATION

Consider the lateral motion of a footbridge which can be approximately represented by Euler-Bernoulli beam equation with viscous damping (Reiner (1947))

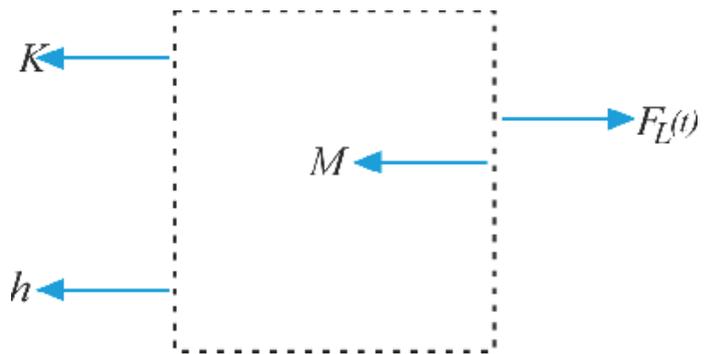


Fig 1: Free Vibration

The free vibration whereby there is no force of pedestrian walking on the footbridges. The assume solution is of the form of series solution and the resulting behavior is described by the fourth order of Partial Differential Equation with viscous damping as:

$$K(x) \frac{\partial^4 u(x,t)}{\partial x^4} + MA \frac{\partial^2 u(x,t)}{\partial t^2} + h \frac{u(x,t)}{\partial t} + T \frac{\partial^2 u(x,t)}{\partial t^2} + Cu = F_L(x,t) \tag{1}$$

where

- $K(x)$ stiffness per unit length [Nm²]
- M mass of beam
- h damping coefficient
- $u(x,t)$ lateral displacement
- A cross section area
- C constant spring
- T compressive force

$$F_L(x,t)=0$$

From the above equations (1) gives the following boundary conditions.

$$U = U(x,t) = \frac{\partial u(x,t)}{\partial x} = 0 \text{ at } x = 0 \text{ or } x = L$$

$$U = U(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} = 0 \text{ at } x = 0 \text{ or } x = L$$

$$U = U(x,t) = \frac{\partial^3 u(x,t)}{\partial x^3} = 0 \text{ at } x = 0 \text{ or } x = L$$

$$U = U(x,t) = \frac{\partial^4 u(x,t)}{\partial x^4} = 0 \text{ at } x = 0 \text{ or } x = L$$

The initial boundary conditions are:

$$U(x,t)=0$$

$$\frac{\partial u(x, 0)}{\partial t} = 0$$

METHOD OF SOLUTION

From equation (1),

$$K(x) \frac{\partial^4 u(x,t)}{\partial x^4} + MA \frac{\partial^2 u(x,t)}{\partial t^2} + h \frac{u(x,t)}{\partial t} + T \frac{\partial^2 u(x,t)}{\partial t^2} + Cu = F_L(x,t) \dots\dots\dots(2)$$

where, $F_L(x,t)=0 \dots\dots\dots(3)$

Dividing the left hand side of equation (2) by K(x) gives

$$\frac{K(x)}{K(x)} \frac{\partial^4 u(x,t)}{\partial x^4} + \frac{MA}{K(x)} \frac{\partial^2 u(x,t)}{\partial t^2} + \frac{h}{K(x)} \frac{u(x,t)}{\partial t} + \frac{T}{K(x)} \frac{\partial^2 u(x,t)}{\partial t^2} + \frac{C}{K(x)} u = 0 \dots\dots\dots 4$$

Assuming $\frac{K(x)}{K(x)}=A, \frac{MA}{K(x)}=B, \frac{h}{K(x)}=C, \frac{T}{K(x)}=D, \frac{C}{K(x)}=E \dots\dots\dots 5$

Substituting equation (5) into equation (4) gives

$$A \frac{\partial^4 u(x,t)}{\partial x^4} + B \frac{\partial^2 u(x,t)}{\partial t^2} + C \frac{u(x,t)}{\partial t} + D \frac{\partial^2 u(x,t)}{\partial t^2} + Eu = 0 \dots\dots\dots 6$$

Rearranging equation (5) becomes

$$A \frac{\partial^4 u(x,t)}{\partial x^4} + (B + D) \frac{\partial^2 u(x,t)}{\partial t^2} + C \frac{u(x,t)}{\partial t} + Eu = 0 \dots\dots\dots 7$$

Using the following assume solution

$$U(x,t) = \sum_{j=1}^{\infty} U(t) \sin\left(\frac{j\pi x}{L}\right) \dots\dots\dots 8$$

equation (7) gives

$$A \left(\frac{j\pi}{L}\right)^4 \sum_{j=1}^{\infty} U(t) \sin\frac{j\pi x}{L} + (B + D) \left(\frac{j\pi}{L}\right)^2 \sum_{j=1}^{\infty} \ddot{U}(t) \sin\frac{j\pi x}{L} + C \sum_{j=1}^{\infty} \dot{U}(t) \sin\frac{j\pi x}{L} + Eu = 0 \dots\dots\dots 9$$

Multiplying equation (6) by $\sin\frac{i\pi x}{L}$ and integrating with respect to x, on the limit of 0 to L.

$$A \sum_{j=1}^{\infty} U(t) \sin\frac{j\pi x}{L} \left(\sin\frac{i\pi x}{L}\right) + (B + D) \sum_{j=1}^{\infty} \ddot{U}(t) \sin\frac{j\pi x}{L} \left(\sin\frac{i\pi x}{L}\right) + C \sum_{j=1}^{\infty} \dot{U}(t) \sin\frac{j\pi x}{L} \left(\sin\frac{i\pi x}{L}\right) + E \sum_{j=1}^{\infty} U(t) \sin\frac{j\pi x}{L} \left(\sin\frac{i\pi x}{L}\right) = 0 \dots\dots 10$$

Using the integration by parts formulae to solve the equation

$$\int_0^L A \sum_{j=1}^{\infty} U(t) \sin\frac{j\pi x}{L} \left(\sin\frac{i\pi x}{L}\right) dx + \int_0^L (B + D) \sum_{j=1}^{\infty} \ddot{U}(t) \sin\frac{j\pi x}{L} \left(\sin\frac{i\pi x}{L}\right) dx + \int_0^L C \sum_{j=1}^{\infty} \dot{U}(t) \sin\frac{j\pi x}{L} \left(\sin\frac{i\pi x}{L}\right) dx + \int_0^L E \sum_{j=1}^{\infty} U(t) \sin\frac{j\pi x}{L} \left(\sin\frac{i\pi x}{L}\right) dx = 0 \dots\dots\dots 11$$

$$A \sum_{j=1}^{\infty} U(t) \int_0^L \sin\frac{j\pi x}{L} \left(\sin\frac{i\pi x}{L}\right) dx + (B + D) \sum_{j=1}^{\infty} \ddot{U}(t) \int_0^L \sin\frac{j\pi x}{L} \left(\sin\frac{i\pi x}{L}\right) dx + C \sum_{j=1}^{\infty} \dot{U}(t) \int_0^L \sin\frac{j\pi x}{L} \left(\sin\frac{i\pi x}{L}\right) dx + E \sum_{j=1}^{\infty} U(t) \int_0^L \sin\frac{j\pi x}{L} \left(\sin\frac{i\pi x}{L}\right) dx = 0 \dots\dots\dots 12$$

$$\sum_{j=1}^{\infty} U^{(iv)}(t) \left[A \int_0^L \sin \frac{j\pi x}{L} \left(\sin \frac{i\pi x}{L} \right) dx + E \int_0^L \sin \frac{j\pi x}{L} \left(\sin \frac{i\pi x}{L} \right) dx \right] \\ + (B + D) \sum_{j=1}^{\infty} \ddot{U}(t) \int_0^L \sin \frac{j\pi x}{L} \left(\sin \frac{i\pi x}{L} \right) dx + C \sum_{j=1}^{\infty} \dot{U}(t) \int_0^L \sin \frac{j\pi x}{L} \left(\sin \frac{i\pi x}{L} \right) dx = 0 \quad \dots\dots\dots 13$$

$$(B + D)\ddot{U}(t) \sum_{j=1}^{\infty} \int_0^L \sin \frac{j\pi x}{L} \left(\sin \frac{i\pi x}{L} \right) dx + C\dot{U}(t) \sum_{j=1}^{\infty} \int_0^L \sin \frac{j\pi x}{L} \left(\sin \frac{i\pi x}{L} \right) dx \\ + (A + E)U(t) \sum_{j=1}^{\infty} \int_0^L \sin \frac{j\pi x}{L} \left(\sin \frac{i\pi x}{L} \right) dx = 0 \quad \dots\dots\dots 14$$

Using the integration by part, assume

$$f(x) = \sin \frac{j\pi x}{L} \quad g'(x) = \sin \frac{i\pi x}{L} \\ f'(x) = \frac{j\pi}{L} \cos \frac{j\pi x}{L} \quad g(x) = -\frac{1}{i\pi} \cos \frac{i\pi x}{L} \quad \dots\dots\dots 15$$

$$\int_0^L \sin \frac{j\pi x}{L} \left(\sin \frac{i\pi x}{L} \right) dx = \sin \frac{j\pi x}{L} \left(-\frac{L}{i\pi} \cos \frac{i\pi x}{L} \right) \Big|_0^L - \int_0^L \frac{j\pi}{L} \cos \frac{j\pi x}{L} \left(\frac{L}{i\pi} \right) \cos \frac{i\pi x}{L} dx \\ \int_0^L \sin \frac{j\pi x}{L} \left(\sin \frac{i\pi x}{L} \right) dx = -\frac{L}{i\pi} \sin \frac{j\pi L}{L} \cos \frac{i\pi L}{L} - \int_0^L \frac{j}{i} \cos \frac{j\pi x}{L} \cos \frac{i\pi x}{L} dx \quad \dots\dots\dots 16$$

Integrating once again

$$f(x) = \cos \frac{j\pi x}{L} \quad g'(x) = \cos \frac{i\pi x}{L} \\ f'(x) = -\frac{j\pi}{L} \sin \frac{j\pi x}{L} \quad g(x) = -\frac{L}{j\pi} \sin \frac{i\pi x}{L} \quad \dots\dots\dots 17$$

$$\int_0^L \cos \frac{j\pi x}{L} \left(\cos \frac{i\pi x}{L} \right) dx = \cos \frac{j\pi x}{L} \left(-\frac{L}{i\pi} \sin \frac{i\pi x}{L} \right) \Big|_0^L - \int_0^L \frac{j\pi}{L} \sin \frac{j\pi x}{L} \left(\frac{L}{i\pi} \right) \sin \frac{i\pi x}{L} dx \\ \int_0^L \cos \frac{j\pi x}{L} \left(\cos \frac{i\pi x}{L} \right) dx = -\frac{L}{i\pi} \cos \frac{j\pi L}{L} \sin \frac{i\pi L}{L} - \int_0^L \frac{j}{i} \sin \frac{j\pi x}{L} \sin \frac{i\pi x}{L} dx \quad \dots\dots\dots 18$$

Substituting equation (18) into equation (16) yields

$$\int_0^L \sin \frac{j\pi x}{L} \left(\sin \frac{i\pi x}{L} \right) dx = -\frac{L}{i\pi} \sin j\pi - \frac{j}{i} \left[-\frac{L}{ix} \cos j\pi \sin i\pi - \int_0^L \frac{j}{i} \sin \frac{j\pi x}{L} \sin \frac{i\pi x}{L} dx \right] \\ \int_0^L \sin \frac{j\pi x}{L} \sin \frac{i\pi x}{L} dx = -\frac{L}{i\pi} \sin j\pi + \frac{jL}{i^2\pi} \cos j\pi \sin i\pi + \frac{j^2}{i^2} \int_0^L \sin \frac{j\pi x}{L} \sin \frac{i\pi x}{L} dx \quad \dots\dots\dots 19$$

$$\left(1 + \frac{j^2}{i^2} \right) \int_0^L \sin \frac{j\pi x}{L} \sin \frac{i\pi x}{L} dx = -\frac{L}{i\pi} \sin j\pi + \frac{jL}{i^2\pi} \cos j\pi \sin i\pi \quad \dots\dots\dots 20$$

$$\int_0^L \sin \frac{j\pi x}{L} \sin \frac{i\pi x}{L} dx = \frac{-\frac{L}{i\pi} \sin j\pi + \frac{jL}{i^2\pi} \cos j\pi \sin i\pi}{\left(1 + \frac{j^2}{i^2} \right)} = K \quad \dots\dots\dots 21$$

$$(B + D)K\ddot{U}(t) + CK\dot{U}(t) + (A + E)KU(t) = 0 \quad \dots\dots\dots 22$$

$$p = \frac{CK}{(B + D)K}$$

$$q = \frac{(A + E)K}{(B + D)K}$$

$$\ddot{U}(t) + p\dot{U}(t) + qU(t) = 0 \dots\dots\dots 23$$

Using characteristics equation
 $U = e^{mt}$

$$m^2 + pm + q = 0 \quad p > q$$

$$m = -p \pm \frac{\sqrt{p^2 - 4q}}{2}$$

$$U(t) = Ae^{m_1 t} + Be^{m_2 t} \dots\dots\dots 24$$

where

$$m_1 = p - \frac{\sqrt{p^2 - 4q}}{2}$$

$$m_2 = -p + \frac{\sqrt{p^2 - 4q}}{2} \dots\dots\dots 25$$

$$U(0)=0$$

$$\frac{du}{dt}(0) = 1$$

$$U(0)=0=A+B \dots\dots\dots 26$$

$$\frac{du(t)}{dt} = -m_1 A e^{-m_1 t} - m_2 B e^{-m_2 t} \dots\dots\dots 27$$

$$\frac{du(0)}{dt} = -m_1 A - m_2 B = 1 \dots\dots\dots 28$$

From equation (26), A=-B
 Substituting equation (27) into equation (28) gives

$$m_1 B - m_2 B = 1$$

$$(m_1 - m_2)B = 1$$

$$B=1 \dots\dots\dots 29$$

$$U(t) = -e^{-m_1 t} + e^{-m_2 t}$$

$$U(x, t) = U(t) \sin \frac{j\pi x}{L}$$

$$U(x, t) = (e^{-m_1 t} + e^{-m_2 t}) \sin \frac{j\pi x}{L} \dots\dots\dots 30$$

RESULTS AND DISCUSSION

The footbridge is not subjected to a force of pedestrian walking on the beam, the equation considered was also forth order partial differential equation (PDE). The PDE was transformed into ordinary differential equation using series solution. The equation was solved analytically using the same assumed solution indicated in equation (8). Computer software was used to analyze the solution obtained graphically with graph plotted in Fig 2 – Fig 4 below. The dynamic response such as the deflection and bending moment of the foot-beam was obtained under different masses M and damping coefficients.

Table 1: Beam Specification

PARAMETERS	VALUES
ε	0.10, 0.11, 0.12
E	0.080, 0.085, 0.09
L	10m
K	2.5e-11N/m ²
c ₀	0, 0.2, 0.5
M	50kg
V	7m/s
I	1
J	2
G	9.8m/s
T	0, 10, 20
H	0, 0.5, 1.0
mA/ms	0.98kg/m ³

Free Vibration on Non-uniform Beam, varying Compressive Force

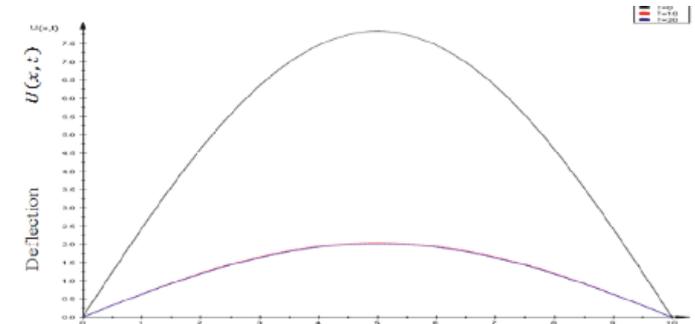


Figure 2: Graph of lateral displacement of non-uniform beam at various values of Compressive Force

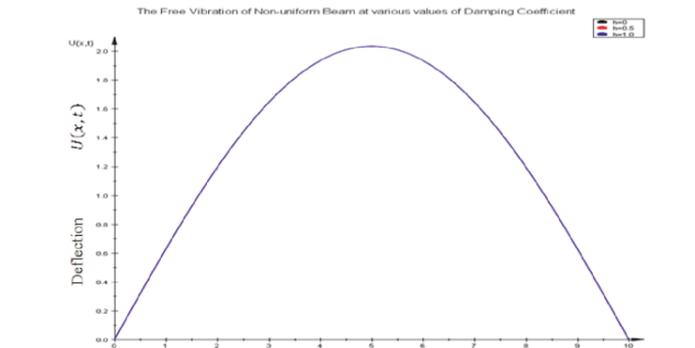


Figure 3: Graph of lateral displacement of non-uniform beam at various values of damping coefficient

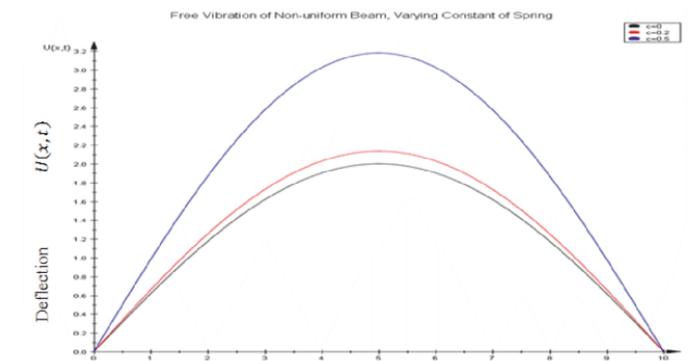


Figure 4: Graph of lateral displacement of non-uniform beam at various values of coefficient of spring constant.

Figure 2 shows the lateral displacement of the non-uniform foot beam reduces for which the compressive force (T) is zero from maximum amplitude of 7.8 to maximum amplitude 2.0 and kept constant for all values of the compressor force other than 0.

From Figure 3, it is observed that the lateral displacement remain the same for all values of the damping coefficient. This implies that the response of the lateral displacement of the non-uniform foot beam is not affected by the damping coefficient.

From figure 4, it is observed that the lateral displacement increases for various values of spring constant.

CONCLUSION

This paper investigates the vibration analysis of footbridges beam. The method of Series solution was used to transform the fourth order partial differential equation to second order Ordinary Differential Equation. The reduced Ordinary Differential Equation was then solved using a numerical method. Numerical simulation was performed to analyze the dynamic response of the

beam with the aid of a computer package (MATLAB) to analyze the solution obtained graphically with the parameters. The dynamic response of the beam was obtained under different damping coefficients, Compressive force, length of the beam, and fixed length of the load. For the free vibration, the result shows that the lateral displacement of the non-uniform beam (footbridges) reduces for which the compressive force (T) is zero from maximum amplitude of 7.8 to maximum amplitude of 2.0 and kept constant for all values of the Compressive Force other than 0. The result also shows that the lateral displacement remain the same for all values of the damping coefficients. This implies that the response of the lateral displacement of the non-uniform beam (footbridge) is not affected by the damping coefficients. The result further shows that the lateral displacement increases for various values of spring constant. From the above results, it can be concluded that there is no noticeable effect of damping in the system as the motion of deflection remains constant for all values of the damping coefficient. Also, the lateral displacement increases for various values of spring constant.

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